Partitioning Multiplex Space into Paths

In this section, we will be using some concepts from abstract algebra. We will see that multiplex space can be partitioned into paths. A path can be thought of as the path of a ball in multiplex space. One might guess that the number of paths determines the number of balls. This is true, but as we will find out later, some of these balls may not be real! In fact, if we allow patterns with throws that are backwards in time, these patterns may involve "antiballs".

Theorem 7
The set of valid patterns together with the operation of composition forms a group.

Proof
The identity is $\mathbf{p}_0(\mathbf{x})=\mathbf{x}$ which maps every element of multiplex space to itself. Composition of functions is an associative operation. Every valid pattern, $\mathbf{p}$, has an inverse because it is a bijection on multiplex space (see section on Multiplex Space).

Definitions:

We define the *positive* $\mathbf{p}$ *path of* $\mathbf{x}$ to be the generated set of $\mathbf{p}$.

$$Path_p^+(\boldsymbol{x}) \equiv \{ \boldsymbol{p}^{\boldsymbol{k}}(\boldsymbol{x}) : \ k \in \mathrm{N} \}$$

We define the *negative* $\mathbf{p}$ *path of* $\mathbf{x}$ to be the generated set of $\mathbf{p}^{-1}$.

$$Path_p^-(\boldsymbol{x}) \equiv \{ \boldsymbol{p}^{-\boldsymbol{k}}(\boldsymbol{x}) : \ k \in \mathrm{N} \}$$

A positive or negative path *diverges to positive infinity* if given any B>0, there is a positive integer T such that when j≥T, the time component of $\mathbf{p}^j(\mathbf{x})$ is greater than B. Similarly, a path *diverges to negative infinity* if given any A<0, there is a negative integer T such that when j≤T, the time component of $\mathbf{p}^j(\mathbf{x})$ is less than A.

Consider $\mathbf{p'}$ to be the permutation on restricted multiplex space (M′) as defined in theorem 2. From abstract algebra, we know that $\mathbf{p'}$ can be partitioned into disjoint cycles. This means that for any member $\mathbf{x}$ of M′, $\mathbf{x}$ is moved by a cycle or is a fixed point. In other words, there exists an integer, l, such that $\mathbf{p'}^l(\mathbf{x}) = \mathbf{x}$.

We define the *orbit* to be

$$O_x \equiv \{ \boldsymbol{p'}^j(\boldsymbol{x}) : \ 0 \le j < l, \ j \in Z \} \ \ where\ \boldsymbol{p'}^l(\boldsymbol{x}) = \boldsymbol{x}.$$

Theorem 8
For a periodic valid pattern, **p**, and for **x** in multiplex space, there exists an integer, l≥1 which satisfies:

$$\boldsymbol{p}^{l}(\boldsymbol{x}) = \boldsymbol{x} + (0,0,L)\cdot\sum_{\boldsymbol{z}\in O_x}\tau(\boldsymbol{x})$$

Proof
From theorem 2,

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}'(\mathbf{x}) + (0,0,\mathrm{L})\cdot\tau(\mathbf{x})$$

Using periodicity,

$$\mathbf{p}^2(\mathbf{x}) = \mathbf{p}(\mathbf{p}'(\mathbf{x}) + (0,0,\mathrm{L})\cdot\tau(\mathbf{x})) = \mathbf{p}(\mathbf{p}'(\mathbf{x})) + (0,0,\mathrm{L})\cdot\tau(\mathbf{x})$$

so

$$\mathbf{p}^2(\mathbf{x}) = \mathbf{p}'^2(\mathbf{x}) + (0,0,\mathrm{L})\cdot(\tau(\mathbf{p}'(\mathbf{x})) + \tau(\mathbf{x})).$$

Proceeding inductively, we find

$$\boldsymbol{p}^{l}(\boldsymbol{x}) = \boldsymbol{p}'^{l}(\boldsymbol{x}) + (0,0,L)\cdot\sum_{\boldsymbol{z}\in O_x}\tau(\boldsymbol{z})$$

where l is the number of elements in the orbit of **x**. Hence, $\mathbf{p}'^{l}(\mathbf{x}) = \mathbf{x}$, and this yields theorem 8.

Definition: We refer to the summation of $\tau(\mathbf{x})$ over the orbit as the *gain* (K) of the positive **p** path of **x**.

Corollary 8A
The positive **p** path of **x** diverges to positive infinity if and only if the negative **p** path of **x** diverges to negative infinity.
Similarly, the positive **p** path of **x** diverges to negative infinity if and only if the negative **p** path of **x** diverges to positive infinity.

Proof
Let K = the gain of the positive **p** path of **x**. It is easily shown by induction on theorem 8 that $\mathbf{p}^{jl}(\mathbf{x}) = \mathbf{x} + jK\cdot(0,0,L)$. Let $\mathbf{x} = (m,h,t)$, $\mathbf{p}^{jl}(\mathbf{x}) = (m_p,h_p,t_p)$, and choose arbitrary bounds A and B. Notice that $t_p = t + jKL$.

If K>0 then choose positive j such that $j > (B-t)/KL$. This implies $t_p > B$. $\mathbf{p}^{jl}(\mathbf{x})$ is in the positive **p** path of **x**. Hence, the positive **p** path of **x** diverges to positive infinity. Now choose negative j such that $j < (A-t)/KL$. This implies $t_p < A$. Hence, the negative **p** path of **x** diverges to negative infinity.

If $K<0$ then choose positive j such that $j > (A-t)/KL$. This implies $jKL < (A-t)$ since K is negative. Hence $t_p < A$ and the positive path diverges to negative infinity. For negative j such that $j < (B-t)/KL$, we get $jKL > (B-t)$, so $t_p > B$ and the negative path diverges to positive infinity.

A path with zero gain is simply an orbit with 1 elements.

Definitions: We define the **p** *path of* **x** to be the union of positive and negative **p** paths of **x** together with **x** itself. The **p** path is *forward* if the positive **p** path diverges to positive infinity. It is *backward* if the positive **p** path diverges to negative infinity. The *gain* of the **p** path equals the gain of its positive **p** path. Given these definitions, it is clear from the proof above that paths with positive gains are forward and paths with negative gains are backward.

Theorem 9
The following statements about a **p** path are equivalent:

(i) A **p** path is an orbit.
(ii) The positive and negative **p** paths do not diverge.
(iii) Its gain is zero.

Proof
We have noted above that **p**'s path forms an orbit if and only if it has zero gain so (i) is equivalent to (iii). We now proceed to show (i) is equivalent to (ii).

Let **x** be some point in multiplex space. Suppose the **p** path of **x** is an orbit. Then the set of elements (m,h,t) of this path is finite. Let B be the maximum value of t for all such elements. Then there is no $t>B$. This implies the positive and negative **p** paths do not diverge to positive infinity. Now let B be the minimum value of t for all such elements. Then there is no $t<B$. This implies the positive and negative **p** paths do not diverge to negative infinity.

Now suppose the positive and negative **p** paths of **x** do not diverge. Let the elements of these paths by denoted (m,h,t). This means there is an upper bound, B, such that no t is greater than B. There is also a lower bound, A, such that no t is less than A. There are $(B-A+1)*H*M$ elements of multiplex space meeting these conditions. Since this number is finite, the path must eventually become cyclic with an orbit having no greater than $(B-A+1)*H*M$ elements.

Theorem 10
The set of **p** paths in multiplex space forms a partition.

Proof
We must show that distinct **p** paths are disjoint and that their union equals multiplex space. Consider the **p** path of **x** and the **p** path of **y**. Suppose that both paths contain the element, **z**. Then there is some integer k such that $\mathbf{z} = \mathbf{p}^k(\mathbf{x})$ and some integer j such that $\mathbf{z} = \mathbf{p}^j(\mathbf{y})$. Solving for **x** gives $\mathbf{x} = \mathbf{p}^{j-k}(\mathbf{y})$ which implies that for any integer, i, $\mathbf{p}^i(\mathbf{x}) = \mathbf{p}^{i+j-k}(\mathbf{y})$. Thus every element of the **p** path of **x** is in the **p** path of **y**. Similarly, we can show that every element of the **p** path of **y** is in the **p** path of **x**. This means the

paths are equal. We have shown two paths are equal if they share an element. The contrapositive of this is that two distinct **p** paths are disjoint. Next, consider a point **z** in multiplex space. **z** is a member of its own **p** path so it is a member of the union of distinct **p** paths. Hence, the union of distinct **p** paths equals multiplex space.

We can partition paths into "spatially distinct" paths. In theorem 8 it was proved that

$$\mathbf{p}^j(\mathbf{x}) = \mathbf{p}'^j(\mathbf{x}) + k_1 \cdot (0,0,L) \qquad (\mathbf{x} \text{ in } M' \text{ and } k_1 \text{ in } Z).$$

Consider any $\mathbf{x}'$ in multiplex space. There is some integer, k_2 such that $\mathbf{x}' = \mathbf{x} + k_2 \cdot (0,0,L)$. By periodicity,

$$\mathbf{p}^j(\mathbf{x}') = \mathbf{p}^j(\mathbf{x} + k_2 \cdot (0,0,L)) = \mathbf{p}'^j(\mathbf{x}) + (k_1 + k_2) \cdot (0,0,L).$$

$$\mathbf{p}^j(\mathbf{x}') - \mathbf{p}'^j(\mathbf{x}) = k \cdot (0,0,L).$$

We say that the elements, $\mathbf{p}'^j(\mathbf{x})$, form an orbit which corresponds with the **p** path of $\mathbf{x}'$. *Spatial equivalence* means equal orbits.

$$\text{Path}_p(\mathbf{x}') \sim \text{Path}_p(\mathbf{y}') \text{ if and only if } O_x = O_y$$

where O_x corresponds with the **p** path of $\mathbf{x}'$ and O_y, with the **p** path of $\mathbf{y}'$. The paths do not necessarily have any elements in M′ but they still correspond to disjoint orbits in M′.

Obviously, this defines an equivalence relation because it satisfies reflexive, symmetric, and transitive properties. We partition paths into spatial equivalence classes.

Theorem 11
$\text{Path}_p(\mathbf{x}') \sim \text{Path}_p(\mathbf{y}')$ if and only if there exists an element **z** of $\text{Path}_p(\mathbf{y}')$ such that $\mathbf{x}' = \mathbf{z} + k \cdot (0,0,L)$ where k is an integer.

Proof
Let **x** be the element of O_x, the corresponding orbit of $\text{Path}_p(\mathbf{x}')$, such that $\mathbf{x}' - \mathbf{x} = k_1 \cdot (0,0,L)$. Since the path of $\mathbf{x}'$ is spatially equivalent to the path of $\mathbf{y}'$ their orbits are equal (i.e. $O_x = O_y$). Hence, there is an integer i such that $\mathbf{p}'_i(\mathbf{y}) = \mathbf{x}$. O_y is a corresponding orbit of $\text{Path}_p(\mathbf{y}')$ implies $\mathbf{p}_i(\mathbf{y}') - \mathbf{p}'_i(\mathbf{y}) = k_2 \cdot (0,0,L)$.

$$\begin{aligned} \mathbf{x}' - \quad \mathbf{x} \quad &= k_1 \cdot (0,0,L) \\ \mathbf{p}_i(\mathbf{y}') - \mathbf{p}'_i(\mathbf{y}) &= k_2 \cdot (0,0,L) \end{aligned}$$

Subtracting these gives

$$\mathbf{x}' - \mathbf{p}_i(\mathbf{y}') = k \cdot (0,0,L).$$

We have found $\mathbf{z} = \mathbf{p}_i(\mathbf{y}')$.

Now suppose we now $\mathbf{x}' = \mathbf{p}_i(\mathbf{y}') + k \cdot (0,0,L)$. Let O_x and O_y be the corresponding orbits of the paths of $\mathbf{x}'$ and $\mathbf{y}'$ respectively. Then $\mathbf{p}_i(\mathbf{y}') = \mathbf{p}'_i(\mathbf{y}) + k_2 \cdot (0,0,L)$. Substituting, we find $\mathbf{x}' = \mathbf{p}'_i(\mathbf{y}) + (k+k_2) \cdot (0,0,L)$. $\mathbf{x}' = \mathbf{x} + k_1 \cdot (0,0,L)$. Substituting this, we get $\mathbf{x} = \mathbf{p}'_i(\mathbf{y}) + (k+k_2-k_1) \cdot (0,0,L)$. Since **x** and $\mathbf{p}'_i(\mathbf{y})$ are in M′, $k+k_2-k_1 = 0$, and $\mathbf{x} = \mathbf{p}'_i(\mathbf{y})$. Therefore, $O_x = O_y$ which means the two paths are spatially equivalent.

Theorem 12
The number of forward **p** paths minus the number of backward **p** paths is equal to the summation of $\tau(\mathbf{x})$ over restricted multiplex space (M′).

Proof
We first proof the following lemma.

Lemma: If K is the gain of a path, $Path_p(\mathbf{x}')$, and K>0 then it belongs to an equivalence class with K members. If K<0 then the equivalence class has −K members. For both cases, these members are $Path_p(\mathbf{x}')$, $Path_p(\mathbf{x}'+(0,0,L))$, $Path_p(\mathbf{x}'+(0,0,2L))$, .. $Path_p(\mathbf{x}'+(0,0,|K|L-L))$.

We must show these members are spatially equivalent and that these represent all paths which are spatially equivalent. By theorem 11, $Path_p(\mathbf{x}') \sim Path_p(\mathbf{x}' + j\cdot(0,0,L))$ for any integer j. So all members are spatially equivalent.

$$Path_p(\boldsymbol{x}) = \{ \boldsymbol{p}^j(\boldsymbol{x}) : \; j \in Z \} = Path_p(\boldsymbol{p}^1(\boldsymbol{x})) = Path_p(\boldsymbol{x}+K\cdot(0,0,L)) .$$

Now consider a spatially equivalent path, $Path_p(\mathbf{x}'+j\cdot(0,0,L))$. Let j=qK+r where $0 \le r < K$ and q,r are integers. Then $Path_p(\mathbf{x}'+j\cdot(0,0,L)) = Path_p(\mathbf{x}'+r\cdot(0,0,L)+qK\cdot(0,0,L)) = Path_p(\mathbf{x}'+r\cdot(0,0,L))$. Since $0 \le r < K$, this is one of the K members of the equivalence class.

Recall that M′ is the union of disjoint orbits, O_x. The union of spatial equivalence classes, each one corresponding to a disjoint orbit, equals the set of **p** paths. Recall that the gain of each path, K, equals the summation of $\tau(\mathbf{x})$ for **x** taken over the corresponding orbit.

$$K = \sum_{\boldsymbol{x} \in O_x} \tau(\boldsymbol{x}) .$$

From our lemma, if K is positive, there are K forward paths in the spatial equivalence class. If K is negative, there are −K backward paths in the spatial equivalence class.

The summation of $\tau(\mathbf{x})$ for **x** in orbits whose corresponding paths each have positive gain is the number of forward **p** paths. Likewise, the summation of $(-\tau(\mathbf{x}))$ for **x** in orbits whose corresponding paths each have negative gain is the number of backward **p** paths.

$$Let\ O_+ \equiv \{ O_x : \; Path_p(\boldsymbol{x}') \text{ corresponds to } O_x \text{ and has } K>0 \}$$

$$Let\ O_- \equiv \{ O_x : \; Path_p(\boldsymbol{x}') \text{ corresponds to } O_x \text{ and has } K<0 \}$$

The union of O_+ and O_- is M′. The summation of $\tau(\mathbf{x})$ for **x** in M′ is therefore

$$N_+ = \sum_{\boldsymbol{x}\in O_-} \tau(\boldsymbol{x}) \qquad N_- = -\sum_{\boldsymbol{x}\in O_-} \tau(\boldsymbol{x})$$

equal to the number of forward paths minus the number of backward paths.

$$N_+ - N_- = \sum_{\boldsymbol{x}\in M'} \tau(\boldsymbol{x})\ .$$

Theorem 13

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = (N_+ - N_-)\cdot(0,0,L)\ .$$

Proof

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in M'} (\boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{x}) = \sum_{\boldsymbol{x}\in M'} (\boldsymbol{p}'(\boldsymbol{x}) + \tau(\boldsymbol{x})\cdot(0,0,L) - \boldsymbol{x})$$

$$= \sum_{\boldsymbol{x}\in M'} \boldsymbol{p}'(\boldsymbol{x}) - \sum_{\boldsymbol{x}\in M'} \boldsymbol{x} + \sum_{\boldsymbol{x}\in M'} \tau(\boldsymbol{x})\cdot(0,0,L) = (N_+ - N_-)\cdot(0,0,L)\ .$$

Notice that if there are no throws backwards in time, there can be no backward paths and the equations becomes

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = (0,0,N_+L)\ .$$

Looking back to theorem 6 we see that $N_+ = N$. Recall that when the concept of a state was introduced, no throws backward in time were allowed. We know that this means there cannot be any backward paths so our definition for N gave the number of forward paths. Theorem 13 is more general. Although mathematically interesting, it is not useful to ordinary jugglers, few of whom can actually throw a ball into the past!! This theorem brings up some interesting questions. Suppose we allow throwing balls into the past as well as into the future. What do N_+ and N_- really represent in terms of numbers of balls being juggled? If a person naturally progressed backward in time, he would throw all balls backward in time. In his view, N_- would represent the number of real balls while N_+ would always be zero. According to Allen Knutson, the constant $(N_+ - N_-)$ represents the number of real balls minus the number of antiballs seen in the pattern at any given moment. This is a very plausible idea. It is no more absurd than the concept of "antimatter", which can be thought of as matter moving backward in time. Let us entertain the question further. Suppose at frame 1 we throw a ball into the future so it lands at frame 2. At this time, we throw the ball back into the past exactly one frame so it lands at frame 1 where it started. Now suppose we observe the pattern. What do we see? At frame 0 nothing has happened so we see nothing. At frame 1, we see two balls suddenly appear. One is a real ball but the other is an antiball. At frame 2 they vanish! This violates our intuitive sense of continuity. We started with no ball; from frame 1 until frame 2 we had two balls; and thereafter we had none again. It is not just the number of balls that must remain constant. It is the difference between real and antiballs which does not change. We see that a ball can be created from nothing if at the same time and place, an antiball is created.

Definitions:
Let t be some real number representing a point in time. Let a ball be thrown from (m_1,h_1,t_1) to (m_2,h_2,t_2), which are points in multiplex space. At time t, the ball is *real* if $t_1 \le t < t_2$. The ball is called an *antiball* if $t_2 \le t < t_1$.

Theorem 14
Considering only the throws in a path, $\mathrm{Path}_p(\mathbf{x})$, the number of real balls minus antiballs seen at any time t is equal to either 1 if that path is forward or −1 if that path is backward.

Proof
Consider a forward path, $\mathrm{Path}_p(\mathbf{x})$. For each an element of the path, $\mathbf{p}^j(\mathbf{x})$, denote its time component by t_j. Let t be some point in time. Because the path is forward, there exists some integer, A, such that for $j \ge A$, $t_j > t$ and there exists some integer, B, such that for $j \le B$, $t_j < t$. If $j \ge A$, balls thrown from $\mathbf{p}^j(\mathbf{x})$ to $\mathbf{p}^{j+1}(\mathbf{x})$ cannot be real balls nor antiballs. Likewise, if $j \le B$, balls thrown from $\mathbf{p}^{j-1}(\mathbf{x})$ to $\mathbf{p}^j(\mathbf{x})$ cannot be real balls nor antiballs. There are exactly (A−B) throws remaining under consideration which take $\mathbf{p}^B(\mathbf{x})$ to $\mathbf{p}^A(\mathbf{x})$. There must be exactly one more of the real balls (N_{real}) than there are of the antiballs (N_{anti}) because of the parity involved in the right and left crossing of time t. This is easily proved using induction. Let C and D be integers such that D−C=1 and consider only the one throw from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^D(\mathbf{x})$. By definition, $N_{real}-N_{anti}$ is 1 if and only if $t_C \le t < t_D$. $N_{real}-N_{anti}$ is −1 if and only if $t_D \le t < t_C$. Furthermore, t_C and t_D are on the same side of t if and only if $N_{real}-N_{anti}$ is zero. Suppose these three statements hold for D−C=K. We show these hold for the case D−C=K+1. This implies D−(C+1)=K.

Case 1. Suppose $t_C \le t < t_D$. If $t_{C+1} > t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is real. Since t_{C+1} and t_D are on the same side of t, $N_{real}-N_{anti}$ is zero considering only these throws. When all throws are considered, N_{real} increases by 1 so $N_{real}-N_{anti}$ is 1. If $t_{C+1} \le t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is neither a real nor an antiball. Since $t_{C+1} \le t < t_D$, $N_{real}-N_{anti}$ is still 1 when all throws are considered.

Case 2. Suppose $t_D \le t < t_C$. If $t_{C+1} \le t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is an antiball. Since t_{C+1} and t_D are on the same side of t, $N_{real}-N_{anti}$ is zero considering only these throws. When all throws are considered, N_{anti} increases by 1 so $N_{real}-N_{anti}$ is −1. If $t_{C+1} > t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is neither a real nor an antiball. Since $t_D \le t < t_{C+1}$, $N_{real}-N_{anti}$ is still −1 when all throws are considered.

Case 3. Now suppose t_C and t_D are both less than or equal to t. If $t_{C+1} \le t$ then no real balls or antiballs are found so $N_{real}-N_{anti}$ remains zero. If $t_{C+1} > t$

we find the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is real. $N_{real}-N_{anti}$ is -1 considering only the other throws. Therefore when all throws are considered, $N_{real}-N_{anti}$ is zero. On the other hand, suppose t_C and t_D are both greater than t. If $t_{C+1}>t$ then no real balls or antiballs are found so $N_{real}-N_{anti}$ remains zero. If $t_{C+1}\leq t$ we find the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is an antiball. $N_{real}-N_{anti}$ is 1 considering only the other throws. $N_{real}-N_{anti}$ is zero when all throws are considered.

The case for a backward **p** path is proved in a similar fashion.

Theorem 15
For any periodic pattern, **p**, and any time t, the number of real balls minus the number of antiballs is equal to $(N_+ - N_-)$.

Proof
The contribution of each forward path to $N_{real}-N_{anti}$ as seen at some time t is 1. The contribution of each backward path to it is -1. Therefore the collection of forward paths (N_+) and backward paths (N_-) contributes (N_+-N_-) to $N_{real}-N_{anti}$. Since these represent a partition of multiplex space, there can be no other contributions to it so N_+-N_- must equal $N_{real}-N_{anti}$.

Extension of the Mathematics to Include Any Pattern

Although the mathematics presented this far is able to describe a large class of juggling patterns, it does not describe all of them. For example, patterns which require the juggler to throw in unusual rythms are not described by the mathematics if the time intervals between throws cannot be represented by integers multiples of some frametime. At the beginning, we stipulated that a pattern consisted of catches and throws which had to occur at discrete points in pattern space. Pattern space was then extended to multiplex space so that the pattern would be a permutation. In reality, time is continuous-- not discretized. All throws and catches in a pattern take place in continuous multiplex space. It is the set of times at which these throws and catches occur which is discrete. We already have developed many theorems that deal with multiplex space. As we shall see, it is not necessary to redevelop these for our new discretized space.

Definition of a Pattern
A pattern, $\mathbf{p}_a$, is a self-mapping of an unbounded countable subset, M_d, of continuous multiplex space (M_c).

Theorem 16
$(\mathbf{p}_a: M_d \rightarrowtail M_d,\ )$ forms a group which is isomorphic to $(\mathbf{p}: M \rightarrowtail M,\ )$.

Proof
Choose some element of M_d and call it t_0. Label all elements by t_j for some integer j so that their order is reflected by their subscripts. (i.e. $t_j>t_k$ implies $j>k$). Let $\mathbf{f}$ map M to M_d by $\mathbf{f}(m,h,j) = (m,h,t_j)$. Note that $\mathbf{f}$ is a bijection because it is both one-to-one and onto. Now let Φ map from $\mathbf{p}_a$ to $\mathbf{p}$: $\mathbf{p} = \Phi(\mathbf{p}_a) = \mathbf{f}^{-1}(\mathbf{p}_a(\mathbf{f}))$. Φ is well-defined because $\mathbf{f}$ is a bijection. Furthermore, Φ is a bijection and we show $\Phi(\mathbf{p}_a\ \mathbf{q}_a) = \Phi(\mathbf{p}_a)\ \Phi(\mathbf{q}_a)$.

$$\Phi(\mathbf{p}_a\ \mathbf{q}_a) = \mathbf{f}^{-1}\ \mathbf{p}_a\ \mathbf{q}_a\ \mathbf{f} = \mathbf{f}^{-1}\ \mathbf{p}_a\ \mathbf{f}\ \mathbf{f}^{-1}\ \mathbf{q}_a\ \mathbf{f} = \Phi(\mathbf{p}_a)\ \Phi(\mathbf{q}_a).$$

What does this mean? Every definition and theorem before was based on **p** and the operation of composition. We have proved this new group, which encompasses all patterns, is isomorphic to our old group. Thus, to analyze any pattern, $\mathbf{p}_a$, we look at $\mathbf{p} = \Phi(\mathbf{p}_a)$. To notate a periodic $\mathbf{p}_a$, it suffices to use MHN in conjunction with the set, $\{t_1, t_2, \ldots\ t_L\}$.

As a side note, it should be mentioned that aperiodic patterns can be viewed as a concatenation of periodic patterns. To link pattern X to pattern Y requires that they have one state in common and the link is made at this state. Thus, the mathematics is capable of describing aperiodic patterns as well as periodic patterns.

A Brief Summary with Conclusions

The axiom of continuity provided a test for pattern validity. It was found that a pattern was represented by a permutation, **p**, on multiplex space (M), an infinite space. Periodicity and restricted multiplex space (M′) were defined and it was determined that for **x** in M′, **p**(**x**) was the sum of **p′**(**x**) and an integer multiple, $\tau(\mathbf{x})$ of (0,0,L) where **p′** was a permutation on M′ (see theorem 2). A way of notating **p** by means of a matrix of vectors was introduced called "Multi-Hand Notation" or "MHN". The operations of permutation, local translation, and global translation were found to preserve validity. The concept of states and transitions was presented and a useful theorem was proved which related the pattern, its period, and the number of balls being juggled (see theorem 6). It was shown that the entire class of patterns together with the operation of composition formed a group. Multiplex space (M) was partitioned into ball paths. It was shown that each path in M corresponded to an orbit, O_x in M′, and also to a gain, K. K was found to equal the summation of $\tau(\mathbf{x})$ for **x** in O_x. Paths which corresponded to the same orbit in M′ were termed spatially equivalent and M was partitioned into spatial equivalence classes. It was proved that each of these spatial equivalence classes had K forward paths if K was positive or −K backward paths if K was negative. The difference between the numbers of forward and backward paths, $(N_+ - N_-)$, was found to be the summation of K for all equivalence classes which was equal to the summation of $\tau(\mathbf{x})$ for **x** in M′ (see theorem 12). A theorem more general than theorem 6 was proved which related a pattern, its period, and the constant, $(N_+ - N_-)$. Antiballs and real balls were defined in terms of being visible at a specific time t. It was determined that the number of real balls minus the number of antiballs was equal to $(N_+ - N_-)$. Finally, a broader definition for a pattern was given which included arhythmic throws. It was determined that any pattern whatsoever could be identified with a rhythmic pattern through an isomorphism, Φ.

Bibliography

Buchthal, David C., and Douglas E. Cameron. *Modern Abstract Algebra*. Boston Mass.: PWS Publishers, 1987.

Magnusson, Bengt, and Bruce Tiemann. *The Physics of Juggling*. The Physics Teacher November 1989, p. 586.

Author's Address

Ed Carstens
RR2 Box 645
Rolla, MO 65401
(314) 364-7536