The Mathematics of Juggling

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11/25/92

ABSTRACT: This paper lays down the foundations for the mathematics of juggling starting with the continuity axiom and defining multi-hand notation, which describes a large class of juggling patterns. Several theorems are stated and proven using these definitions and the axiom of continuity. One need not know how to juggle to appreciate these ideas. The mathematics is interesting by itself and can be appreciated by the non-juggler as well as the juggler.

Introduction

Juggling is an art form which has been around for ages. There is much disagreement over the definition of juggling. Nowadays, the world of juggling includes cigar box manipulation, diablo, shaker cups, devil sticking, and even contact juggling, an art form popularized by Michael Moschen. For the purposes of this article, juggling shall be defined as keeping a number of objects in motion. This definition does not mention how objects are moved. In fact, it does not require any of the motions be periodic or even that hands be used to catch and throw the objects.

When there is repetition to the movements of objects, this is called a periodic pattern. The most widely known three-ball pattern is the cascade, in which the juggler alternates right- and left-hand throws in a regular rhythm making each throw the same height. There are thousands of patterns which involve alternating right- and left-hand throws in a regular rhythm. These patterns are called siteswaps and they lend themselves very nicely to a concise notation. (In siteswap notation, the three-ball cascade is simply 3.) Siteswap notation is the most widely known notation among jugglers today. It was invented by Paul Klymack from Santa Cruz. Siteswaps are briefly explained in an article, "The Physics of Juggling" by Bengt Magnusson and Bruce Tiemann (The Physics Teacher November 1989 p.586).

In the summer of 1991, I invented multi-hand notation (MHN), which is a concise and mathematical way of describing patterns involving any number of hands and not necessarily throwing in an alternating manner. This notation, which was originally presented to the usenet group, rec.juggling, is not yet as well-known as siteswap notation, however I believe it is indispensable as a tool for mathematical analysis of juggling. I also wrote a program called JugglePro which notates patterns in MHN and using high-resolution graphics, juggles balls on-screen according to the indicated pattern. This program is even capable of moving the hands while juggling and catching and throwing more than one ball at a time with one hand.

Representation of Pattern Space and Patterns

The *pattern space* in which juggling takes place is represented by (h,t) where h and t represent the hand and time at which a throw or catch occurs. In other words, throws and catches are events which take place at discrete points in (h,t) space. This is defined as

$$\wp \equiv \{(h,t) : 1 \le h \le H \quad h,t \in Z\}$$

where H represents the number of hands and Z is the set of integers.

The juggler needs a set of instructions which describe where each ball is to be thrown. This is accomplished by means of a vector function, $\mathbf{p}_m$, which takes a point (h,t) and gives that throw's destination, $\mathbf{p}_m(h,t)$, which is another point in pattern space. The juggler may have more than one ball in one hand at a time. If there are M balls at (h,t), then the subscript m ranges from 1 to M so that there is a different function for each ball. The set of functions, $\{\mathbf{p}_m: 1 \le m \le M\}$, represents the *pattern*.

Equation of Continuity in Pattern Space

In this section, the simple notion of continuity is put into precise mathematical terms. To do this some definitions must be made first. When a ball is caught it is usually held for some time before it is thrown again. This time is called the *hold time*. In a mathematical treatment of a juggling pattern it is helpful to suppress the notion of hold time by letting it be zero. (There is no loss in generality in doing this.) A *frame* is any point in time where catches and throws occur (simultaneously). A *frametime* is the time interval between successive frames. For any particular frame and hand, the number of balls entering or being caught by the hand must equal the number of balls leaving or being thrown by that hand. This is the axiom of continuity. Expressed mathematically we have,

$$\sum_{\boldsymbol{x} \in \wp} \sum_{m=1}^{M} \delta(\boldsymbol{z} - \boldsymbol{p}_m(\boldsymbol{x})) = M \qquad (\forall \boldsymbol{z} \in \wp)$$

The equation tells us that if we choose any point, $\mathbf{z}$, in the pattern space, and we count the total number of throws, $\mathbf{p}_m(\mathbf{x})$ to that position, we get M, the number of throws being made from point $\mathbf{z}$. Patterns with M>1 have a special name given to them since they require the juggler to be able to catch and throw more than one ball using the same hand. They are called multiplex patterns.

Patterns are *valid* if they satisfy the equation of continuity. It is interesting to note, however, that a valid pattern is not necessarily physically realizable. A valid pattern can have throws which move the ball backwards in time! We will not deal with these rather abstract patterns until later. Therefore we add the jugglability condition - a pattern is *jugglable* if it is valid and has no throws backwards in time and if there is a throw with zero time component, it must be a self-throw. The latter statement takes care of the "rest" which occurs in a frame when a hand has no ball to catch.

Periodicity

The general equation of continuity is of little practical help because it requires an infinite summation. In practice, we encounter *periodic* patterns, which repeat after a certain number of frames. Therefore, we confine our interest to this class. Suppose a pattern repeats after L frames. We say its period is L and since it is periodic we have

$$\mathbf{r}_m(h,t) = \mathbf{r}_m(h,t+L)$$

where $\mathbf{r}_m(h,t)$ is the displacement vector representing the M throw(s) from position (h,t) and is given by

$$\mathbf{r}_m(h,t) = \mathbf{p}_m(h,t) - (h,t).$$

Given that the pattern is periodic, we can restrict (h,t) to one period of pattern space. It is sufficient to know only one period of the pattern in order to know the entire pattern. We define the period space to be

$$\mathfrak{L} \equiv \{(h,t): 1 \le h \le H,\ 1 \le t \le L\}$$

The equivalent definition of periodicity for the positional form of the pattern can now be found. If a pattern P is periodic with period L, then

$$\boldsymbol{p}_{\boldsymbol{m}}(h,t) + k(0,L) = \boldsymbol{p}_{\boldsymbol{m}}(h,t+kL) \quad \forall k \in Z$$

The pattern can be represented by an HxLxM matrix of vectors, $\mathbf{p}_m(h,t)$. This brings us to multi-hand notation (MHN), which is devised to represent periodic patterns.

Multi-Hand Notation (MHN)

A throw takes an object (a ball) to a new location in (h,t) space. This destination vector is placed at the location in (h,t) space from which the throw initiates. A matrix of vectors describes the throws making up the pattern. This is the *positional form* of the pattern. The subscript m is used for multiplex patterns. For M>1, a different 2-d matrix would be needed for each m=1,2,...M. Thus, the multiplex pattern matrix becomes three dimensional.

$$P = \begin{bmatrix} \boldsymbol{p}_m(1,1) & \boldsymbol{p}_m(1,2) & \ldots & \boldsymbol{p}_m(1,L) \\ \boldsymbol{p}_m(2,1) & & & \\ . & & & \\ . & & & \\ \boldsymbol{p}_m(H,1) & \boldsymbol{p}_m(H,2) & \ldots & \boldsymbol{p}_m(H,L) \end{bmatrix}$$

The zero permutation matrix, P_0, takes each (h,t) location to itself.

$$P_0 = \begin{bmatrix} (1,1) & (1,2) & \ldots & (1,L) \\ (2,1) & & & \\ . & & & \\ . & & & \\ (H,1) & (H,2) & \ldots & (H,L) \end{bmatrix}$$

The relative form of a pattern is found by subtracting P_0 from P.

$$R \equiv P - P_0$$

As we shall see later, P_0, is the identity for composition of permutations.

Equation of Continuity for Periodic Patterns

In this section we develop a method of verifying continuity for a periodic pattern which involves a summation over just one period. Again the concept is to count the number of balls being thrown to a particular hand in a frame and check to see if this count equals M, the number of balls being thrown from the hand. If we restrict our count to just one period, we will miss those balls whose throws originate outside this period. How do we account for these? Consider a throw that originates in the previous period. The concept of periodicity tells us that the same throw takes place exactly L frametimes later, which is in the period of interest. Therefore, we count this as one of the balls being thrown to the hand of interest.

Theorem 1

$$\sum_{\boldsymbol{x} \in \mathfrak{L}} \sum_{m=1}^{M} \delta\left(\boldsymbol{z} - \boldsymbol{p}_m'(\boldsymbol{x})\right) = M \qquad \forall \boldsymbol{z} \in \wp$$

where we define $\mathbf{p}_m'(\mathbf{x})$ as follows:

Let $(h_p, t_p) = \mathbf{p}_m(\mathbf{x})$ and let $(h_z, t_z) = \mathbf{z}$.

$$\boldsymbol{p}_m'(\boldsymbol{x}) \equiv \boldsymbol{p}_m(\boldsymbol{x}) - \left(0, [\![(t_p - t_z)/L]\!] \cdot L\right)$$

Proof
We start with the general equation of continuity and sum it period by period.

$$\sum_{\tau=-\infty}^{\infty} \sum_{t=1-\tau L}^{L-\tau L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta\left(\boldsymbol{z} - \boldsymbol{p}_m(h, t)\right) = M$$

Applying the definition of periodicity to the equation of continuity, we obtain:

$$\sum_{\tau=-\infty}^{\infty} \sum_{t=1-\tau L}^{L-\tau L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta\left(\boldsymbol{z} - \boldsymbol{p}_m(h, t+\tau L) + (0, \tau L)\right) = M$$

Now we change the order of the sum and let $t' = t + \tau L$.

$$\sum_{t'=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{\tau=-\infty}^{\infty} \delta\left(\boldsymbol{z} - \boldsymbol{p}_m(h, t') + (0, \tau L)\right) = M$$

There is but one integer τ which can possibly satisfy:

$$\boldsymbol{z} = \boldsymbol{p}_m(h, t') - (0, \tau L).$$

It is

$$\tau' = [\![(t_p - t_z)/L]\!]$$

Thus, we can drop the summation over τ by replacing $\mathbf{p}_m$ by $\mathbf{p}_m'$ as defined in theorem 1. When we do this, we get equation of Theorem 1.

Multiplex Space

So far, we have been denoting throw-vectors by $\mathbf{p}_m(h,t)$. The subscript m can take on integer values from 1 to M, the multiplex limit. The multiplex limit is the number of balls which can be caught and thrown from one hand at one time. We have been considering these M different throws occuring at (h,t) as being interchangeable. From now on, however, we will identify each throw with a number from 1 to M. We begin by defining *multiplex space*.

$$\mathrm{M} \equiv \{(m,h,t): 1 \le m \le M,\ 1 \le h \le H,\ -\infty < t < \infty\}$$

We can think of a single vector function, **p**, mapping multiplex space to pattern space. This function is not one-to-one because there are M vectors, **x** = (m,h,t), which all map to the same position for each position in pattern space. We would like to have a one-to-one function. We proceed by adding one more dimension to the range of **p**, making it be a one-to-one mapping from multiplex space to multiplex space.

For each **z** = (h,t) in pattern space, there are M vectors **x** in multiplex space which map to **z** under the function **p**. Denote these vectors $\mathbf{x}_m$ where m goes from 1 to M. Redefine **z** = (m,h,t). Now **p** maps multiplex space to multiplex space. Each **x** is mapped to one **z** by the function **p** because **p** is single-valued. For each **z** = (m,h,t), there exists but one **x** for which **p**(**x**) = **z**. Therefore, **p** is a bijection.

For multiplex space, the equation of continuity becomes

$$\sum_{\boldsymbol{x} \in \mathrm{M}} \delta(\boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{z}) = 1 \quad \forall \boldsymbol{z} \in \mathrm{M}.$$

When restricted to a period, the equation can be stated as

$$\sum_{\boldsymbol{x} \in \mathrm{M}'} \delta(\boldsymbol{p}'(\boldsymbol{x}) - \boldsymbol{z}) = 1 \quad \forall \boldsymbol{z} \in \mathrm{M}'.$$

where

$$\mathrm{M}' \equiv \{\ (m,h,t):\ 1 \le m \le M,\ 1 \le h \le H,\ 1 \le t \le L\ \}$$

and where both the domain and range of **p′** is restricted to one period of multiplex space in such a way that

$$\boldsymbol{p}'(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}) - (0,0,\tau(\boldsymbol{x}) \cdot L) \quad \textit{where } \tau(\boldsymbol{x}) \textit{ is an integer}.$$

The equation of continuity shows that **p′** is a bijection and therefore it is a permutation on restricted multiplex space. We have proven a valuable theorem about periodic patterns. It allows one to quickly generate hosts of valid patterns by simple permutations and additions.

Theorem 2.

$$P = perm(P_0^{\prime}) + (0,0,L) \cdot E.$$

where E is the excitation matrix of integers

$$E = \begin{bmatrix} \tau(m,1,1) & \tau(m,1,2) & \ldots & \tau(m,1,L) \\ \tau(m,2,1) & \tau(m,2,2) & \ldots & \tau(m,2,L) \\ . & & & \\ . & & & \\ \tau(m,H,1) & \tau(m,H,2) & \ldots & \tau(m,H,L) \end{bmatrix}.$$

E is really a three-dimensional matrix whose elements, $\tau(\mathbf{x})$, are the integers used in defining $\mathbf{p}'(\mathbf{x})$. Theorem 2 defines $\mathbf{p}$ only in restricted multiplex space. The domain of $\mathbf{p}$ can be extended to multiplex space by using the fact that $\mathbf{p}$ is periodic. Any element of multiplex space can be written as $(\mathbf{x} + k(0,0,L))$ where $\mathbf{x}$ is an element of restricted multiplex space and k is an integer.

$$\mathbf{p}(\mathbf{x} + k(0,0,L)) = \mathbf{p}(\mathbf{x}) + k(0,0,L) = \mathbf{p}'(\mathbf{x}) + (0,0,L) \cdot (\tau(\mathbf{x})+k).$$

Theorem 2 can be stated in words: Any valid periodic pattern is representable by the sum of some permutation matrix and (0,0,L) times some excitation matrix.

Pattern Operations which Preserve Validity

There are three useful operations which can be easily proven to preserve validity. They are permutation, local translation, and global translation. Consider an arbitrary permutation on restricted multiplex space, $\mathbf{q}(\mathbf{x})$. Suppose that $\mathbf{p}$ is a valid pattern. $\mathbf{q}(\mathbf{x})$ is in restricted multiplex space so Theorem 2 says:

$$\mathbf{p}(\mathbf{q}(\mathbf{x})) = \mathbf{p}'(\mathbf{q}(\mathbf{x})) + (0,0,L) \cdot \tau(\mathbf{q}(\mathbf{x}))$$

$\mathbf{p}'(\mathbf{q})$ is a permutation on restricted multiplex space and $\tau(\mathbf{q}(\mathbf{x}))$ is defined and is an integer. As done before, we extend the domain of $\mathbf{p}(\mathbf{q})$ to multiplex space by using the fact that it is periodic and this gives a valid pattern.

A local translation is defined as an addition of any integer number times L to any one of the time components of the throw-vectors in the matrix. It is also easily proven from Theorem 2.

$$P + (0,0,L) \cdot E_{local} = perm(P_0^{\prime}) + (0,0,L) \cdot (E_{local} + E)$$

The new excitation matrix for (P + (0,0,L)E_{local}) is (E + E_{local}). Hence, the local translation of P is also valid.

A global translation is defined as the addition of some integer, g, to the time components of all of the vectors in the matrix. It is proven using the general equation of continuity.

$$\sum_{\boldsymbol{x}\in \mathrm{M}} \delta\left(\boldsymbol{p}(\boldsymbol{x}) - [\boldsymbol{z}-(0,0,g)]\right) = 1 \qquad \forall \boldsymbol{z}\in \mathrm{M}.$$

$$\sum_{\boldsymbol{x}\in \mathrm{M}} \delta\left([\boldsymbol{p}(\boldsymbol{x})+(0,0,g)]-\boldsymbol{z}\right) = 1 \qquad \forall \boldsymbol{z}\in \mathrm{M}.$$

The elements of the globally translated matrix are $\mathbf{p}(\mathbf{x})+(0,0,g)$. Therefore, global translation preserves validity.

States and Transitions

So far we have only looked at patterns by themselves. That is, we have not thought about what a juggler might do to go from pattern X to pattern Y. Our present notation does not tell us where all the balls are at every frame of the pattern. It only tells us what throws to make at each frame. To denote the configuration of balls in (h,t) space we define a *state matrix*. It should be noted that the state matrix as defined here is only for those valid patterns whose throws are all forward into time.

$$S_0 = \begin{bmatrix} s(1,1) & s(1,2) & \ldots & s(1,W) \\ s(2,1) & s(2,2) & \ldots & s(2,W) \\ & . & & \\ s(H,1) & s(H,2) & \ldots & s(H,W) \end{bmatrix}$$

$$W \equiv \max t_r \quad \textit{where we let } (h_r, t_r) = r_m(x) \qquad (x\in\mathfrak{L})$$

This matrix is HxW where W, the *width* of the matrix, is the maximum throw height. Each element tells how many balls there are at that location in (h,t) space. Elements beyond this width, W, are all taken to be zero. This *state space* is a subset of pattern space and is defined

$$\sigma = \{(h,t):\ 1\le h\le H,\ \ 0\le t<\infty\}$$

As time passes, each ball moves to the left in the matrix. To be precise, after one frametime each element gets shifted one column to the left. Then each hand makes new throws prescribed by the pattern matrix and the state matrix becomes

$$S_1 = \begin{bmatrix} s(1,2) & s(1,3) & \dots & s(1,W) & 0 \\ s(2,2) & s(2,3) & \dots & s(2,W) & 0 \\ . & & & & \\ s(H,2) & s(H,3) & \dots & s(H,W) & 0 \end{bmatrix} + S_1'$$

where the components of S'_t are defined as

$$s_t'(i,j) \equiv \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,t) - (0,t) - (i,j))$$

This is what is called a *transition*. The fundamental transition equation is

$$S_{a+1} = D(S_a) + S_{a+1}'$$

where a is any integer and $D(S_a)$ is the decay of S_a by one frametime (i.e. left-shift). Expressed in terms of components we have

$$s_{a+1}(i,j) = s_a(i,j+1) + \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,a+1) - (0,a+1) - (i,j))$$

Theorem 3
For any integer $t \geq 1$,

$$s_{a+t}(i,j) = s_a(i,j+t) + \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+a+t))$$

Proof
From the fundamental transition equation, we have inductively that

$$S_{a+t} = D^{(t)}(S_a) + D^{(t-1)}(S_{a+1}') + D^{(t-2)}(S_{a+2}') + \dots + S_{a+t}'$$

Stating this equation in terms of components we have

$$s_{a+t}(i,j) = s_a(i,j+t) + \sum_{h=1}^{H} \sum_{m=1}^{M} [\ \boldsymbol{\delta}(\boldsymbol{p}_m(h,a+1)-(0,a+1)-(i,j+t-1))$$

$$+ \boldsymbol{\delta}(\boldsymbol{p}_m(h,a+2)-(0,a+2)-(i,j+t-2)) + \ldots$$

$$+ \boldsymbol{\delta}(\boldsymbol{p}_m(h,a+t)-(0,a+t)-(i,j))\ ]$$

These sums can now be combined under a third summation. When this is done, we get Theorem 3.

Corollary 3A

For any integer $t \geq W$,

$$s_{a+t}(i,j) = \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \boldsymbol{\delta}(\boldsymbol{p}_m(h,l) - (i,j+a+t))$$

Proof
This is obvious because $s_a(i,j+t) = 0$ for $t \geq W$.

A periodic pattern of period L goes through L states. In the ith frame, the throws would be given by the ith column of the pattern matrix. The state matrix for the ith frame would be S_i. One might say at first that there is a flaw in this definition because it requires that you first know the previous state. However, it is clear from Corollary 3A that after $t \geq W$ frames all of the elements of S_a have been shifted out so that S_{a+t} no longer depends on S_a but only on the pattern matrix, P. Furthermore, if the pattern matrix, P, is completely known, then any state in the pattern can be found directly from P. This must be true since a can be chosen arbitrarily.

The periodicity of the pattern implies that states must repeat after L frames. We prove this theorem next.

Theorem 4

$$S_t = S_{(t+L)}$$

Proof
We start with a statement of Corollary 3A.

$$s_{t+L}(i,j) = \sum_{l=t+L-W}^{t+L} \sum_{h=1}^{H} \sum_{m=1}^{M} \boldsymbol{\delta}(\boldsymbol{p}_m(h,l)-(i,j+t+L))$$

From the definition of periodicity we get,

$$= \sum_{l=t+L-W}^{t+L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l-L) - (i,j+t))$$

$$= \sum_{l=t-W}^{t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(p_m(h,l) - (i,j+t)) = s_t(i,j).$$

Theorem 5
For a valid pattern (one which satisfies the continuity equation) the components of S are bounded by M.

Proof
From Corollary 3A we have

$$s_{a+t}(i,j) = \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+a+t)) \quad \text{for } t \geq W$$

From the general equation of continuity we have

$$M = \sum_{l=-\infty}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+a+t)) \geq s_{a+t}(i,j)$$

Intuitively, we know that a component of S must not exceed the multiplex limit, M. M represents the total number of balls one hand can handle at one time. If s(i,j) were to exceed M, then when it came time to catch the s(i,j) balls, there would be too many to handle.

The following theorem is an extension of Shannon's Theorem. It relates the number of balls, N, to the period, L, and the relative pattern matrix, R.

Theorem 6

$$\sum_{\boldsymbol{x} \in \mathrm{M}'} \boldsymbol{r}(\boldsymbol{x}) = (0,0,NL)$$

where

$$N = \sum_{\boldsymbol{x} \in \sigma} s_a(\boldsymbol{x}) \qquad \forall a.$$

Proof
From Theorem 2 we have,

$$P = perm(P_0^{\prime}) + (0,0,L)\cdot E.$$

where P is the pattern matrix of vectors (m,h,t).

Summing over restricted multiplex space and applying the commutative property of addition to perm(P'_0),

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{p}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in M'} \boldsymbol{p}_0^{\prime}(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in M'} (0,0,L)\cdot \tau(\boldsymbol{x})$$

Since $\mathbf{p}'_0(\mathbf{x}) = \mathbf{x}$,

$$\sum_{\boldsymbol{x}\in M} (\boldsymbol{p}(\boldsymbol{x})-\boldsymbol{x}) = \sum_{\boldsymbol{x}\in M} \boldsymbol{r}(\boldsymbol{x}) = (0,0,L)\cdot\sum_{\boldsymbol{x}\in M} \tau(\boldsymbol{x}).$$

This proves the theorem for the first two components. I have not found a simple proof for the last component. We start with the transition theorem for a=0 and t=L. ($\mathbf{p}_m$ maps pattern space into pattern space.)

$$s_L(i,j) = s_0(i,j+L) + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+L))$$

Now we multiply both sides by j and form a sum over state space.

$$\sum_{i=1}^{H} \sum_{j=1}^{\infty} js_L(i,j) = \sum_{i=1}^{H} \sum_{j=1}^{\infty} js_0(i,j+L) + term2$$

$$term2 = \sum_{i=1}^{H} \sum_{j=1}^{\infty} j \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(\boldsymbol{x}) - (i,j+L))$$

$$term1 = \sum_{h=1}^{H} \sum_{j=1}^{\infty} j\ s_0(i,j+L) = \sum_{h=1}^{H} \sum_{j=1}^{\infty} (j+L-L)\ s_0(i,j+L)$$

$$= \sum_{i=1}^{H} \sum_{j=1+L}^{\infty} j \; s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1+L}^{\infty} L \; s_0(i,j)$$

$$= \left[\sum_{i=1}^{H} \sum_{j=1}^{\infty} j \; s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1}^{L} j \; s_0(i,j) \right]$$

$$- \; L \left[\sum_{i=1}^{H} \sum_{j=1}^{\infty} s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1}^{L} s_0(i,j) \right]$$

$$= \left[\sum_{h=1}^{H} \sum_{j=1}^{\infty} j \; s_0(i,j) \right] - L \cdot N + \sum_{i=1}^{H} \sum_{j=1}^{L} (L-j) \; s_0(i,j) .$$

We now define the *excitation value*, XV, of a state, S.

$$XV(S) \equiv \sum_{i=1}^{H} \sum_{j=1}^{\infty} j \; s(i,j) .$$

Our total equation now becomes

$$XV(S_L) = XV(S_0) - LN + \sum_{i=1}^{H} \sum_{j=1}^{L} (L-j) s_0(i,j) + term2$$

We know that since $S_0 = S_L$, $XV(S_0) = XV(S_L)$. Thus, the equation becomes

$$NL = \sum_{j=1}^{L} \sum_{i=1}^{H} (L-j) s_0(i,j) + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=1}^{\infty} j \; \delta(p_m(h,l) - (i,j+L)) .$$

$$term2 = \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=1}^{\infty} j \delta(\boldsymbol{p_m}(h,l) - (i,j+L))$$

Collecting terms we have,

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \left[\sum_{j=-\infty}^{\infty} j\delta(\boldsymbol{p}_m(h,l) - (i,j+L)) - \sum_{j=-\infty}^{0} j\delta(\boldsymbol{p}_m(h,l) - (i,j+L)) \right]$$

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_p - L) - \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=-\infty}^{L} (j - L)\delta(\boldsymbol{p}_m(h,l) - (i,j))$$

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_R + l - L) + \sum_{j=1}^{L} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p}_m(h,l) - (i,j))$$

$$+ \sum_{j=-\infty}^{0} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p}_m(h,l) - (i,j))$$

$$\sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_R(h,l) + l - L) = \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) + \sum_{j=1}^{L} \sum_{i=1}^{H} \sum_{m=1}^{M} (j - L).$$

$$NL = \sum_{j=1}^{L} \sum_{i=1}^{H} (L - j)\left[s_0(i,j) - M + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j)) \right]$$

$$+ \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) + \sum_{j=-\infty}^{0} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p}_m(h,l) - (i,j))$$

For the first term, we make the following substitutions for $s_0(i,j)$ and M:

$$s_0(i,j) = \sum_{l=-\infty}^{0} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j)) \quad (\textit{Corollary 2A})$$

$$M = \sum_{l=-\infty}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j)) \qquad (\textit{continuity})$$

The equation reduces to

$$NL = \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) - \sum_{j=1}^{L} \sum_{i=1}^{H} (L - j) \sum_{l=L+1}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j))$$

$$+\sum_{j=-\infty}^{0}\sum_{i=1}^{H}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{M}(L-j)\,\delta(\boldsymbol{p}_m(h,l)-(i,j))$$

The last two terms are both identically zero. The reason is that throws cannot be made backwards in time. This means

$$t_R(\boldsymbol{x}) \geq 0 \quad \Rightarrow \quad t_p(\boldsymbol{x}) \geq t_x$$

where $\mathbf{p}_m(\mathbf{x}) = (h_p(\mathbf{x}), t_p(\mathbf{x}))$,

$\mathbf{r}_m(\mathbf{x}) = (h_R(\mathbf{x}), t_R(\mathbf{x}))$,

and $\mathbf{x} = (h_x, t_x)$.

In the second to last sum, the condition of jugglability implies $t_p(h,l) \geq l$. Clearly, $j<l$ in this sum. Hence, $j<t_p(h,l)$. Since j never equals $t_p(h,l)$, the total sum is zero. Likewise, in the very last sum, $j<l\leq t_p(h,l)$. Therefore its total is also zero. That leaves the equation,

$$NL = \sum_{\boldsymbol{x}\in\mathfrak{L}}\sum_{m=1}^{M} t_R(\boldsymbol{x}) \,.$$

This concludes the proof of Theorem 6. As a corollary, we add the obvious conclusion that the number of balls, N, is equal to the sum of elements of the excited state matrix, E.

Corollary 6A

$$N \equiv \sum_{\boldsymbol{x}\in\sigma} s_a(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in\mathrm{M}'} \tau(\boldsymbol{x}) \,.$$